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On Atypical Values and Local Monodromies of Meromorphic Functions

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Abstract—A meromorphic function on a compact complex analytic manifold defines a C^∞ locally trivial fibration over the complement to a finite set in the projective line $\mathbb{CP}^1$—the bifurcation set. Loops around points of the bifurcation set give rise to corresponding monodromy transformations of this fibration. We show that the zeta-functions of these monodromy transformations can be expressed in local terms, namely, as integrals of zeta-functions of meromorphic germs with respect to the Euler characteristic. A particular case of meromorphic functions on the projective space $\mathbb{CP}^n$ are those defined by polynomial functions of n variables. We describe some applications of this technique to polynomial functions.

1. INTRODUCTION

We want to consider fibrations defined by meromorphic functions. In order to have more general statements we prefer to use the notion of a meromorphic function slightly different from the standard one. Let M be an n-dimensional compact complex analytic manifold.

Definition. A *meromorphic function* f on the manifold M is a ratio $\frac{P}{Q}$ of two nonzero sections of a line bundle $\mathcal{L}$ over M. Meromorphic functions $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ (where P' and Q' are sections of a line bundle $\mathcal{L}'$) are *equal* if $P = U \cdot P'$ and $Q = U \cdot Q'$ where U is a section of the bundle $\mathrm{Hom}(\mathcal{L}', \mathcal{L}) = \mathcal{L} \otimes \mathcal{L}'^*$ without zeros (in particular, this implies that the bundles $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic).

A particular important case of meromorphic functions are rational functions $\frac{P(x_0,\dots,x_n)}{Q(x_0,\dots,x_n)}$ on the projective space $\mathbb{CP}^n$ (P and Q are homogeneous polynomials of the same degree).

A meromorphic function $f = \frac{P}{Q}$ defines a map f from the complement $M \setminus \{P = Q = 0\}$ of the set of common zeros of P and Q to the complex line $\mathbb{CP}^1$. The indeterminacy set $\{P = Q = 0\}$ may have components of codimension one. For $c \in \mathbb{CP}^1$, let $F_c = f^{-1}(c)$.

The standard arguments (using a resolution of singularities; see, e.g., [7, 8]) give the following statement.

Theorem 1. *The map $f\colon M \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a C^∞ locally trivial fibration outside a finite subset of the projective line $\mathbb{CP}^1$.*

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Any fiber $F_{\text{gen}} = f^{-1}(c_{\text{gen}})$ of this fibration is called a generic fiber of the meromorphic function f. The smallest subset $B(f) \subset \mathbb{CP}^1$ for which f is a C^∞ locally trivial fibration over $\mathbb{CP}^1 \setminus B(f)$ is called the *bifurcation set* of the meromorphic function f. Its elements are called *atypical values* of the meromorphic function f.

Remark 1. In addition to some other advantages (see Statement 1 and Remark 6) the described definition of a meromorphic function permits treatment in the same way of the following situation. Let $f = \frac{P}{Q}$ be a meromorphic function on the manifold M and let $H = \{R = 0\}$ be a hypersurface in M (R is a section of a line bundle). One can be interested to study the map defined by the restriction of the meromorphic function f to $M \setminus (H \cup \{P = Q = 0\})$. Substituting the meromorphic function f by $f' = \frac{PR}{QR}$ one reduces the situation to the discussed one, i.e., the indeterminacy set of the meromorphic function f' coincides with $H \cup \{P = Q = 0\}$ and the meromorphic functions f and f' coincide outside it.

A loop in the complement $\mathbb{CP}^1 \setminus B(f)$ to the bifurcation set $B(f)$ gives rise to a monodromy transformation of the fiber bundle. The monodromy transformation is defined only up to homotopy (or rather up to isotopy), but the monodromy operator (the action of the monodromy transformation in the homotopy groups of the generic fiber of the meromorphic function f) is well defined. Therefore the fundamental group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ of the complement to the bifurcation set acts on the homology groups $H_*(F_{\text{gen}}; \mathbb{C})$ of the generic fiber of the meromorphic function f. The image of the group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ in the group of automorphisms of $H_*(F_{\text{gen}}; \mathbb{C})$ is called the *monodromy group* of the meromorphic function f. It is generated by local monodromy operators corresponding to simple loops around the atypical values of f (see [2]).

For a map $h\colon X \to X$ of a topological space X (say, with finite-dimensional homology groups) into itself, its zeta-function $\zeta_h(t)$ is the rational function defined by

$$\zeta_h(t) = \prod_{q \geq 0} \{\det[\mathrm{id} - t\, h_*|_{H_q(X;\mathbb{C})}]\}^{(-1)^q}.$$

Remark 2. This is the definition of the zeta-function of a map from [2]. The zeta-function defined in [1] is the inverse of this one.

Let $\zeta_f^c(t)$ be the zeta-function of the local monodromy corresponding to the value $c \in \mathbb{CP}^1$ (i.e., defined by a simple loop around c).

Remark 3. Local monodromy and the corresponding zeta-function are defined for any value $c \in \mathbb{CP}^1$, not only for atypical ones. For a generic value of the meromorphic function f, the local monodromy is the identity and its zeta-function is equal to $(1 - t)^{\chi(F_{\text{gen}})}$.

2. GERMS OF MEROMORPHIC FUNCTIONS AND THEIR INVARIANTS

A meromorphic function $f = \frac{P}{Q}$, its monodromy transformations and the corresponding zeta-functions are, in some sense, "global objects." In this paper we reduce the problem of computing the zeta-functions of local monodromies to local computations. E.g., the computation of the zeta-function of the local monodromy around $0 \in \mathbb{CP}^1$ is reduced to computations at points of the zero-locus $\{P = 0\}$ of (the section) P. At a point of $\{P = 0\} \setminus \{P = Q = 0\}$ the function f determines a germ of a holomorphic function, whence at a point of the indeterminacy set $\{P = Q = 0\}$ it determines a germ of a meromorphic function. Therefore the discussed computations are reduced to computations (of appropriate zeta-functions) for holomorphic and meromorphic germs. Those for holomorphic germs have a long history and are more known. Let us recall basic notions and facts concerning meromorphic germs from [5] (with slight alternations).

A *germ of a meromorphic function* on $(\mathbb{C}^n, 0)$ is a fraction $f = P/Q$, where P and Q are germs of holomorphic functions $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Two germs of meromorphic functions $f = P/Q$ and $f' = P'/Q'$ are said to be *equal* if $P' = U \cdot P$ and $Q' = U \cdot Q$ for a germ of a holomorphic function $U\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ with $U(0) \neq 0$.

For $c \in \mathbb{CP}^1$ the *c-Milnor fiber of the germ* f is the set [a (noncompact in general) $(n-1)$-dimensional complex manifold with boundary]

$$\mathcal{M}_f^c = \{z \in B_\varepsilon\colon\ f(z) = P(z)/Q(z) = c'\}$$

for ε small enough (depending on c) and c' close enough to c (depending on ε; if $c = \infty$, "c' close to c" means that $\|c'\|$ is large enough). In the above definition B_ε stands for the closed ball of radius ε with center at the origin in $\mathbb{C}^n$. It is possible to show that this notion is well defined.

Remark 4. It can be easily seen that $\mathcal{M}_f^c = \mathcal{M}_{f-c}^0$ for $c \neq \infty$, where $f - c = \frac{P-cQ}{Q}$, and $\mathcal{M}_f^\infty = \mathcal{M}_{1/f}^0$, where $\frac{1}{f} = \frac{Q}{P}$. If $Q(0) \neq 0$, the meromorphic germ f is in fact holomorphic. In this case its c-Milnor fiber coincides with the usual Milnor fiber (of a holomorphic germ) for $c = f(0)$ and it is empty for $c \neq f(0)$.

One can show that, for ε small enough, the sets $\{z \in B_\varepsilon\colon\ f(z) = P(z)/Q(z) = c'\}$ are fibers of a locally trivial fibration over a punctured neighborhood of the point c. Therefore there is defined (up to isotopy) the monodromy transformation $h_f^c\colon \mathcal{M}_f^c \to \mathcal{M}_f^c$ which corresponds to a loop going around the point c in the positive direction (i.e., counter-clockwise). The c-monodromy operator is the (well-defined) action of the corresponding monodromy transformation in a homology group of the Milnor fiber. The c-zeta-function $\zeta_f^c(t)$ of the meromorphic germ f is the zeta-function of the c-monodromy transformation $h_f^c\colon \mathcal{M}_f^c \to \mathcal{M}_f^c$.

There are two main methods of calculation of the zeta-function of the classical monodromy transformation for a holomorphic germ. One of them is based on the formula of N. A'Campo [1] which expresses the zeta-function in terms of a resolution of the germ. The other one uses the formula of A.N. Varchenko [8] which expresses the zeta-function of a holomorphic germ in terms of its Newton diagram. It can be used for a germ non-degenerate with respect to the Newton diagram.

For a germ of a meromorphic function it is convenient to write an analog of the formula of N. A'Campo for two zeta-functions: for the 0- and the ∞-ones (the calculation of the c-zeta-function $\zeta_f^c(t)$ of a germ f for $c \neq 0, \infty$ can be reduced to the calculation of the 0-zeta-function of the germ $f - c$).

A *resolution* of a meromorphic germ $f = P/Q$ is a modification of the space $(\mathbb{C}^n, 0)$ (i.e., a proper analytic map $\pi\colon \mathcal{X} \to \mathcal{U}$ of a smooth analytic manifold $\mathcal{X}$ onto a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$, which is an isomorphism outside of a proper analytic subspace in $\mathcal{U}$) such that the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ is a normal crossing divisor at each point of the manifold $\mathcal{X}$. The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings implies that, in a neighborhood of any point of it, there exists a local system of coordinates $y_1, y_2, \ldots, y_n$ such that the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of the functions P and Q to the space $\mathcal{X}$ of the resolution are equal to $u \cdot y_1^{k_1} y_2^{k_2} \cdot \ldots \cdot y_n^{k_n}$ and $v \cdot y_1^{\ell_1} y_2^{\ell_2} \cdot \ldots \cdot y_n^{\ell_n}$, respectively, where $u(0) \neq 0$ and $v(0) \neq 0$, k_i and ℓ_i are nonnegative integers.

Let the resolution $\pi\colon \mathcal{X} \to \mathcal{U}$ of the germ f be an isomorphism outside the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ and let $\mathcal{D} = \pi^{-1}(0)$ be the preimage of the origin of $\mathbb{C}^n$ ($\mathcal{D}$ also is a normal crossing divisor). Let $S_{k,\ell}$ be the set of points of the divisor $\mathcal{D}$ in a neighborhood of which in some local coordinates the liftings $P \circ \pi$ and $Q \circ \pi$ of the functions P and Q have the form $u \cdot y_1^k$ and $v \cdot y_1^\ell$, respectively, with $u(0) \neq 0$, $v(0) \neq 0$. Then

$$\zeta_f^0(t) = \prod_{k>\ell}(1 - t^{k-\ell})^{\chi(S_{k,\ell})}, \qquad \zeta_f^\infty(t) = \prod_{k<\ell}(1 - t^{\ell-k})^{\chi(S_{k,\ell})}.$$

Remark 5. A resolution π of the germ $f' = RP/(RQ)$ is at the same time a resolution of the germ $f = P/Q$. Moreover the multiplicities of any component C of the exceptional divisor in the zero divisors of the liftings $(RP) \circ \pi$ and $(RQ) \circ \pi$ of the germs RP and RQ are obtained from those for the germs P and Q by adding one and the same integer—the multiplicity $m = m(C)$ of the component C in the zero divisor of the lifting of the germ R. Nevertheless the meromorphic germs f and f' may have different zeta-functions. The reason why the above formulas may give different results for germs f and f' consists in the fact that if an open part of the component C lies in $S_{k,\ell}(f)$ then, generally speaking, its part which lies in $S_{k+m,\ell+m}(f')$ is smaller.

The Varchenko type formula for meromorphic germs can be also found in [5].

3. ZETA-FUNCTIONS OF LOCAL MONODROMIES

Let $f = \frac{P}{Q}$ be a meromorphic function on the complex manifold M.

The following statement is a direct consequence of the definitions.

Statement 1. *Let $\pi\colon \widetilde{M} \to M$ be an analytic map of an n-dimensional compact complex manifold $\widetilde{M}$ which is an isomorphism outside of the union of the indeterminacy set $\{P = Q = 0\}$ of the meromorphic function f and of a finite number of level sets $f^{-1}(c_i)$. Let $\widetilde{f} = \frac{P\circ\pi}{Q\circ\pi}$ be the lifting of the meromorphic function $f = \frac{P}{Q}$ to $\widetilde{M}$. Then the generic fiber of $\widetilde{f}$ coincides with that of f and for each $c \in \mathbb{CP}^1$ one has*

$$\zeta_{\widetilde{f}}^c(t) = \zeta_f^c(t).$$

Remark 6. Even if the indeterminacy set $\{P = Q = 0\}$ of the meromorphic function f has codimension two (i.e., if the hypersurfaces $\{P = 0\}$ and $\{Q = 0\}$ have no common components), in general, this is not the case for the lifting $\widetilde{f}$. This is a reason for our definition of a meromorphic function. If one starts from the usual definition, the lifting $\widetilde{f}$ of the meromorphic function f can be defined at some points of the preimage $\pi^{-1}(\{P = Q = 0\})$ of the indeterminacy set. In this case a generic level set of the meromorphic function $\widetilde{f}$ differs from that of f and Statement 1 does not hold. The simplest example is the function $f = \frac{x}{y}$ in affine coordinates on the plane $\mathbb{CP}^2$, π being the blowing-up of the origin in this affine chart.

In order to have somewhat more attractive and unified formulas we would like to use the notion of the integral with respect to the Euler characteristic [9]. The main property of a usual (say, Lebesgue) measure, which, together with the positivity condition, permits definition of the notion of the integral, is the property $\sigma(X \cup Y) = \sigma(X) + \sigma(Y) - \sigma(X \cap Y)$. The Euler characteristic possesses this property too. In this sense it can be considered as a measure, though nonpositive. Nonpositivity of the Euler characteristic imposes restrictions on the class of functions for which the integral with respect to the Euler characteristic can be defined.

Let A be an Abelian group with the group operation $*$, and let X be a semianalytic subset of a complex manifold. Let $\Psi\colon X \to A$ be a function on X with values in A for which there exists a finite partitioning $\mathcal{S}$ of X into semianalytic sets (strata) Ξ such that the function Ψ is constant on each stratum Ξ (and equal to ψ_Ξ). Then by definition the integral with respect to the Euler characteristic of the function Ψ over the set X is equal to

$$\int_X \Psi(x)\,d\chi = \sum_{\Xi\in\mathcal{S}} \chi(\Xi)\psi_\Xi,$$

where $\chi(\Xi)$ is the Euler characteristic of the stratum Ξ. In the above formula we use the additive notations for the operation $*$. In what follows this definition will be used for integer-valued functions and also for local zeta-functions $\zeta_x(t)$ which are elements of the Abelian group of nonzero rational functions in the variable t with respect to multiplication. In the last case in the multiplicative notations the above formula means

$$\int_X \zeta_x(t)\,d\chi = \prod_{\Xi\in\mathcal{S}} (\zeta_\Xi(t))^{\chi(\Xi)}.$$

Let c be a point of the projective line $\mathbb{CP}^1$. For a point $x \in M$, let $\zeta^c_{f,x}(t)$ be the corresponding zeta-function of the germ of the meromorphic function f at the point x and let $\chi^c_{f,x}$ be its degree $\deg \zeta^c_{f,x}(t)$.

Theorem 2.

$$\zeta^c_f(t) = \int_{\{P=Q=0\}\cup F_c} \zeta^c_{f,x}(t)\,d\chi, \tag{1}$$

$$\chi(F_{\text{gen}}) - \chi(F_c) = \int_{F_c} (\chi^c_{f,x} - 1)\,d\chi + \int_{\{P=Q=0\}} \chi^c_{f,x}\,d\chi. \tag{2}$$

Proof follows the lines of the proof of Theorem 1 in [4]. Without any loss of generality one can suppose that $c = 0$. There exists a modification $\pi\colon \mathcal{X} \to M$ of the manifold M which is an isomorphism outside the set $\{P = Q = 0\} \cup \{f = 0\} \cup \{f = \infty\} = \{P = 0\} \cup \{Q = 0\}$ such that $\mathcal{D} = \pi^{-1}(\{P = 0\} \cup \{Q = 0\})$ is a normal crossing divisor in the manifold $\mathcal{X}$. Then at each point of the exceptional divisor $\mathcal{D}$ in a local system of coordinates one has $P \circ \pi = u \cdot y_1^{k_1} \cdot \ldots \cdot y_n^{k_n}$, $Q \circ \pi = v \cdot y_1^{\ell_1} \cdot \ldots \cdot y_n^{\ell_n}$ with $u(0) \neq 0$, $v(0) \neq 0$, $k_i \geq 0$ and $\ell_i \geq 0$. There exist Whitney stratifications $\mathcal{S}$ and $\mathcal{S}^*$ of M and $\mathcal{X}$, respectively, such that:

(1) the map π is a stratified morphism with respect to these stratifications;

(2) the set $\{P = 0\} \cup \{Q = 0\}$ is a stratified subspace of the stratified space $(M, \mathcal{S})$;

(3) for each stratum $\Xi^* \in \mathcal{S}^*$ the germs of the liftings $P \circ \pi$ and $Q \circ \pi$ of the sections P and Q at points of Ξ^* have normal forms $u \cdot y_1^{k_1} \cdot \ldots \cdot y_n^{k_n}$ and $v \cdot y_1^{\ell_1} \cdot \ldots \cdot y_n^{\ell_n}$, where $(k_1, \ldots, k_n)$ and $(\ell_1, \ldots, \ell_n)$ do not depend on a point of Ξ^*;

(4) for each stratum $\Xi \in \mathcal{S}$ the zeta-function $\zeta^c_{f,x}(t)$ does not depend on the point x for $x \in \Xi$.

Remark 7. Actually the point (4) follows from the first three ones. However it is convenient to include it in the list of conditions.

One applies the following version of the formula of A'Campo [1] and also its local variant for meromorphic germs (Section 2). Let $X_{k,\ell}$ be the set of points of the manifold $\mathcal{X}$ in the neighborhood of which the liftings $P \circ \pi$ and $Q \circ \pi$ of P and Q in some local coordinates have the forms $u \cdot y_1^k$ and $v \cdot y_1^\ell$, respectively ($u(0) \neq 0$, $v(0) \neq 0$).

Statement 2.

$$\zeta_f^0(t) = \prod_{k>\ell\geq 0} (1 - t^{k-\ell})^{\chi(X_{k,\ell})}.$$

The property (1) of the stratifications $\mathcal{S}$ and $\mathcal{S}^*$ implies that the morphism π is locally trivial over each stratum of $\mathcal{S}$: if the stratum Ξ of $\mathcal{S}$ is the image of the stratum Ξ^* of $\mathcal{S}^*$, $\Xi = \pi(\Xi^*)$, then $\pi\colon \Xi^* \to \Xi$ is a smooth locally trivial fiber bundle. In particular

$$\chi(\Xi^*) = \chi(\Xi) \cdot \chi(\pi^{-1}(x) \cap \Xi^*), \qquad x \in \Xi.$$

Let $\mathcal{S}_{k,\ell}$ be the set of strata from $\mathcal{S}^*$ such that the germs of the liftings $P \circ \pi$ and $Q \circ \pi$ of P and Q at their points are equivalent to y_1^k and y_1^ℓ, respectively; $X_{k,\ell} = \bigcup_{\Xi^* \in \mathcal{S}_{k,\ell}} \Xi^*$. We have

$$\begin{aligned}
\zeta_f^0(t) &= \prod_{k>\ell\geq 0} (1 - t^{k-\ell})^{\chi(X_{k,\ell})} = \prod_{k>\ell\geq 0} (1 - t^{k-\ell})^{\sum_{\Xi^* \in \mathcal{S}_{k,\ell}} \chi(\Xi^*)} \\
&= \prod_{k>\ell\geq 0} \prod_{\Xi^* \in \mathcal{S}_{k,\ell}} (1 - t^{k-\ell})^{\chi(\Xi^*)} = \prod_{k>\ell\geq 0} \prod_{\Xi \in \mathcal{S}} \prod_{\Xi^* \in \mathcal{S}_{k,\ell} \cap \pi^{-1}(\Xi)} (1 - t^{k-\ell})^{\chi(\Xi^*)} \\
&= \prod_{\Xi \in \mathcal{S}} \prod_{k>\ell\geq 0} \prod_{\Xi^* \in \mathcal{S}_{k,\ell} \cap \pi^{-1}(\Xi)} (1 - t^{k-\ell})^{\chi(\Xi) \cdot \chi(\pi^{-1}(x) \cap \Xi^*)} \\
&= \prod_{\Xi \in \mathcal{S}} \left(\prod_{k>\ell\geq 0} \prod_{\Xi^* \in \mathcal{S}_{k,\ell} \cap \pi^{-1}(\Xi)} (1 - t^{k-\ell})^{\chi(\pi^{-1}(x) \cap \Xi^*)} \right)^{\chi(\Xi)} \\
&= \prod_{\Xi \in \mathcal{S}} [\zeta_{f,x}^0(t)]^{\chi(\Xi)} = \int_{\{P=Q=0\} \cup F_0} \zeta_{f,x}^0(t)\, d\chi.
\end{aligned}$$

As usual the formula for the Euler characteristic of the generic fiber follows from the formula for the zeta-function, since it is the degree of the latter.

The difference between $(\chi_{f,x}^c - 1)$ and $\chi_{f,x}^c$ in the two integrals in (2) reflects the fact that the Euler characteristic of the local level set $F_c \cap B_\varepsilon(x)$ ($B_\varepsilon(x)$ is the ball of small radius ε centered at the point x) of the germ of the function f is equal to 1 for a point x of the level set F_c and is equal to 0 for a point x of the indeterminacy set $\{P = Q = 0\}$. In the first case this local level set is contractible and in the second one it is the difference between two contractible sets. □

Let us denote $(-1)^{n-1}$ times the first and the second integrals in (2) by $\mu_f(c)$ and $\lambda_f(c)$, respectively. Let $\mu_f = \sum_{c\in\mathbb{CP}^1} \mu_f(c)$, $\lambda_f = \sum_{c\in\mathbb{CP}^1} \lambda_f(c)$ (in each sum only finite number of summands are different from zero).

Theorem 3.

$$\mu_f + \lambda_f = (-1)^{n-1}\Big(2 \cdot \chi(F_{\text{gen}}) - \chi(M) + \chi(\{P = Q = 0\})\Big).$$

Proof. One has

$$\int_{\mathbb{CP}^1} \chi(F_c)\, d\chi = \chi(M \setminus \{P = Q = 0\}) = \chi(M) - \chi(\{P = Q = 0\}).$$

Therefore

$$\chi(M) - \chi(\{P = Q = 0\}) = \int\limits_{\mathbb{CP}^1} \chi(F_{\text{gen}})\, d\chi + \int\limits_{\mathbb{CP}^1} (\chi(F_c) - \chi(F_{\text{gen}}))\, d\chi$$

$$= 2 \cdot \chi(F_{\text{gen}}) - (-1)^{n-1} \sum_{c \in \mathbb{CP}^1} (\mu_f(c) + \lambda_f(c)) = 2 \cdot \chi(F_{\text{gen}}) + (-1)^n(\mu_f + \lambda_f). \qquad \square$$

Let $\widetilde{f}$ be the restriction of f to $M \setminus \{Q = 0\}$, $\widetilde{f}\colon M \setminus \{Q = 0\} \to \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$. Notice that the fibers of both maps f and $\widetilde{f}$ over values $c \in \mathbb{C}$ coincide.

Corollary 1.

$$\chi(F_{\text{gen}}) = \chi(M) - \chi(\{Q = 0\}) + (-1)^{n-1}(\lambda_f - \lambda_f(\infty) + \mu_f - \mu_f(\infty)).$$

Let f be the meromorphic function on the complex projective space $\mathbb{CP}^n$ defined by a polynomial P in n variables (see below). If P has only isolated critical points in $\mathbb{C}^n$, then $\mu_f(c)$ is the sum of the Milnor numbers of the critical points of the polynomial P with critical value c and $\lambda_f(c)$ is equal to the invariant $\lambda_P(c)$ studied in [3]. Therefore $\mu_f(c)$ and $\lambda_f(c)$ can be considered as generalizations of those invariants (they have sense also in the case when critical points of the polynomial P are not isolated). One has $\mu_f = \mu_P + \mu_f(\infty)$, $\lambda_f = \lambda_P + \lambda_f(\infty)$, where $\mu_P = \sum_{c \in \mathbb{C}} \mu_P(c)$, $\lambda_P = \sum_{c \in \mathbb{C}} \lambda_P(c)$. Notice that in this case Corollary 1 turns into the well-known formula $\chi(F_{\text{gen}}) = 1 + (-1)^{n-1}(\lambda_P + \mu_P)$.

4. APPLICATIONS TO POLYNOMIALS

A polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ defines a meromorphic function $f = \frac{\widetilde{P}}{x_0^d}$ on the projective space $\mathbb{CP}^n$ ($d = \deg P$). For any $c \in \mathbb{CP}^1$, the local monodromy of the polynomial P and its zeta-function $\zeta_P^c(t)$ are defined (in fact they coincide with those of the meromorphic function f). The described technique gives the following statements for polynomials. Let us remember that for $x \in \{P = c\} \subset \mathbb{C}^n$ the zeta-function $\zeta_{f,x}^c(t)$ is the usual zeta-function $\zeta_{P,x}^c(t)$ of the germ of the polynomial P at x.

Theorem 4. *For* $c \in \mathbb{C} \subset \mathbb{CP}^1$,

$$\zeta_P^c(t) = \left(\int\limits_{\{\widetilde{P}=0\} \cap \mathbb{CP}^{n-1}_\infty} \zeta_{f,x}^c(t)\, d\chi \right) \left(\int\limits_{\{P=c\}} \zeta_{P,x}^c(t)\, d\chi \right). \tag{3}$$

For the infinite value,

$$\zeta_P^\infty(t) = \int\limits_{\mathbb{CP}^{n-1}_\infty} \zeta_{f,x}^\infty(t)\, d\chi.$$

For a generic $c' \in \mathbb{C}$,

$$\chi(\{P = c'\}) - \chi(\{P = c\}) = \int\limits_{\{\widetilde{P}=0\} \cap \mathbb{CP}^{n-1}_\infty} \chi_{f,x}^c\, d\chi + \int\limits_{\{P=c\}} (\chi_{P,x}^c - 1)\, d\chi.$$

Remark 8. In [6] we consider the zeta-function of the local monodromy (corresponding to a finite value c) of the polynomial P near infinity which is just the first factor in formula (3). If that zeta-function is different from 1, then the value c is atypical at infinity.

Let $H = \{R = 0\}$ be a hypersurface in $\mathbb{C}^n$ ($R\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial). The polynomial P restricted to the complement of the hypersurface H defines a C^∞ locally trivial fibration over the complement to a finite subset of $\mathbb{C}$. For each value $c \in \mathbb{C}$ as well as for $c = \infty$, the local monodromy of this fibration and its zeta-function $\zeta^c_{P/H}(t)$ are defined. The described fibration is nothing else but the fibration defined by the meromorphic function $\widetilde{f} = \frac{P \cdot R}{R}$. It implies the following result.

Theorem 5. *For* $c \in \mathbb{C} \subset \mathbb{CP}^1$,

$$\zeta^c_{P/H}(t) = \int\limits_{\{P=c\}\cup(\{\widetilde{P}=0\}\cap\mathbb{CP}^{n-1}_\infty)} \zeta^c_{\widetilde{f},x}(t)\, d\chi.$$

For a generic $c' \in \mathbb{C}$,

$$\chi(F_{c'} \setminus H) - \chi(F_c \setminus H) = \int\limits_{\{\widetilde{P}=0\}\cap\mathbb{CP}^{n-1}_\infty} \chi^c_{\widetilde{f},x}\, d\chi + \int\limits_{\{P=c\}\cap H} \chi^c_{\widetilde{f},x}\, d\chi + \int\limits_{F_c\setminus H} (\chi^c_{P,x} - 1)\, d\chi.$$

Now let P and Q be two polynomials in n variables $x_1, \ldots, x_n$ of degree d_1 and d_2, respectively. Without loss of generality we may assume that $d_1 \geq d_2$. For $z = (z_1 : z_2) \in \mathbb{CP}^1$, let $X_z = \{x \in \mathbb{C}^n : z_1 P(x) + z_2 Q(x) = 0\}$. $\Lambda = \{X_z\}_{z\in\mathbb{CP}^1}$ is a pencil of affine hypersurfaces. Let $X = \{(z,x) = (z_1 : z_2, x) \in \mathbb{CP}^1 \times \mathbb{C}^n : z_1 P(x) + z_2 Q(x) = 0\} \subset \mathbb{CP}^1 \times \mathbb{C}^n$ and let $\pi\colon X \to \mathbb{CP}^1$ be the projection to the first factor. There exists the smallest finite set $\Sigma \subset \mathbb{CP}^1$ such that the projection $\pi\colon X \setminus \pi^{-1}(\Sigma) \to \mathbb{CP}^1 \setminus \Sigma$ is a C^∞ locally trivial fibration. The fiber X_{gen} of this fibration is called the generic fiber of the pencil Λ. Let $Y = \{P = Q = 0\} \subset \mathbb{C}^n$ be the base set of the pencil Λ.

For $c \in \mathbb{CP}^1$, let $h^c\colon X_{\mathrm{gen}} \to X_{\mathrm{gen}}$ be the monodromy transformation corresponding to a simple loop around the value c and let $\zeta^c_\pi(t)$ be the zeta-function of the monodromy transformation h^c. One can choose h^c in such a way that h^c is a homeomorphism of X_{gen}, $h^c_{|Y} = \mathrm{id}$. Then

$$\zeta^c_\pi(t) = \zeta_{h^c|Y}(t) \cdot \zeta_{h^c|(X_{\mathrm{gen}},Y)}(t)$$

(see [2]). One has

$$\zeta_{h^c|Y}(t) = (1-t)^{\chi(Y)} = \int\limits_Y (1-t)\, d\chi.$$

Let $\widetilde{P}(x_0, x_1, \ldots, x_n) = x_0^{d_1} P(x_1/x_0, \ldots, x_n/x_0)$ and $\widetilde{Q}(x_0, x_1, \ldots, x_n) = x_0^{d_2} Q(x_1/x_0, \ldots, x_n/x_0)$ be the homogenized polynomials of P and Q and let

$$f(x_0 : x_1 : \ldots : x_n) = \frac{\widetilde{P}(x_0, x_1, \ldots, x_n)}{x_0^{d_1-d_2}\widetilde{Q}(x_0, x_1, \ldots, x_n)}$$

be a meromorphic function on $\mathbb{CP}^n$.

For $z \in \mathbb{CP}^1$, one has $X_z \setminus Y = f^{-1}(z)$; the monodromy transformation $h^c_{\pi|X_{\mathrm{gen}}\setminus Y}$ coincides with $h^c_f\colon f^{-1}(z_{\mathrm{gen}}) \to f^{-1}(z_{\mathrm{gen}})$. Theorem 2 gives

$$\zeta_{h^c|(X_{\mathrm{gen}},Y)}(t) = \int\limits_{X_c\cup\mathbb{CP}^{n-1}_\infty} \zeta^c_{f,x}(t)\, d\chi.$$

Thus we have

Theorem 6.

$$\zeta^c_\pi(t) = \int\limits_{(X_c\setminus Y)\cup\mathbb{CP}^{n-1}_\infty} \zeta^c_{f,x}(t)\, d\chi \cdot \int\limits_Y (1-t)\zeta^c_{f,x}(t)\, d\chi.$$

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